Chap IV Lebesgue Integral

§1.

Here we only give a short summary (details are set as ex.)

Let $\varphi \in \mathcal{S}_0$ (simple function vanishing outside a set of finite mea),
with its canonical representation

$$\varphi = \sum_{j=1}^{N} b_j \chi_{\varphi^{-1}(b_j)} = \sum_{j=0}^{N} b_j \chi_{\varphi^{-1}(b_j)}$$

(assuming φ <u>not</u> constantly zero) : —just, in particular, $m(\varphi^{-1}(b_j)) < \infty$

Define the integrals

$$\int \varphi := \sum_{j=1}^{N} b_j\, m(\varphi^{-1}(b_j)) = \sum_{j=0}^{N} b_j\, m(\varphi^{-1}(b_j)) \quad \Big(\text{convention } 0\cdot\infty = 0\Big)$$

and

$$\int_E \varphi := \int (\varphi \chi_E) = \sum_{j=1}^{N} b_j\, m(E \cap \varphi^{-1}(b_j)) \quad \Big((\varphi\chi_E)^{-1}(b_j) = E \cap \varphi^{-1}(b_j)\Big)$$

($\forall E \in \mathfrak{M}$).

(All functions below are from $\mathcal{S}_0$)

Facts: 1) If $\varphi_1 \sim \varphi_2$ ($\varphi_1 = \varphi_2$ a.e) then $\int \varphi_1 = \int \varphi_2$

2) $\varphi \mapsto \int \varphi$ is linear and $\uparrow$ ($\varphi_1 \leq \varphi_2$ a.e $\Rightarrow \int \varphi_1 \leq \int \varphi_2$)

3) $A \mapsto \int_A \varphi$ is countably additive

4) Assume $\varphi \geq 0$ a.e (so $\int \varphi \geq 0$). Then $\int \varphi = 0 \Leftrightarrow \varphi = 0$ a.e.

(pf. Under the stated assumption, $0 = \int \varphi = \sum b_j\, m(\varphi^{-1}(b_j)) \Rightarrow b_j\, m\varphi^{-1}(b_j) = 0\ \forall j$

5) $\varphi_1 = \varphi_2$ on E ($\Rightarrow \int_E \varphi_1 = \int_E \varphi_2$)

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§2 $\int_E f$ with $m(E) < +\infty$ & f bounded, say $m \leq f(x) \leq M\ \forall x$.

Let $\mathcal{S}(E)$ $\overset{m(E)<+\infty}{}$ denote the set of all simple functions on E (or on $\mathbb{R}$ vanishing outside E) — so $\int_E \varphi$ already defined, $\forall \varphi \in \mathcal{S}(E)$

Define, $\forall$ bounded function f on E,

$$\overline{\int_E} f = \inf\{\int_E \psi : \psi \in \mathcal{S}(E), f \leq \psi \text{ on } E\}$$

$$\underline{\int_E} f = \sup\{\int_E \varphi : \varphi \in \mathcal{S}(E), \varphi \leq f \text{ on } E\}$$

($f \leq \psi$ on E can be replaced by $f \leq \psi$ a.e on E)

Notes.

1. $m \leq \underline{\int_E} f \leq \overline{\int_E} f \leq M$ (where m, M are lower/upper bds of f)

2. $\underline{\int_E} f$ & $\overline{\int_E} f$ $\uparrow$ (as f "increases"); in particular

   $0 \leq f$ a.e on $E \Rightarrow 0 \leq \underline{\int_E} f, \underline{\int_E} f$

3. Sublinear & superlinear resp. for $\overline{\int_E}$ & $\underline{\int_E}$

4. $-\underline{\int_E} f = \overline{\int_E} f$ (as $-\sup(A) = \inf(-A)$ $\forall A \subseteq \mathbb{R}$)

5. When $E = [a,b]$,

   $$(\mathcal{R})\underline{\int_a^b} f \leq (\mathcal{L})\underline{\int_E} f \leq (\mathcal{L})\overline{\int_E} f \leq (\mathcal{R})\overline{\int_a^b} f$$

   (since step $\Rightarrow$ simple)

   Cor. If f is Riemann-integrable on $[a,b]$ then $(\mathcal{L})\underline{\int_E} f = (\mathcal{L})\overline{\int_E} f$.

Th 1. Let $m(E) < +\infty$ and $f : E \to \mathbb{R}$ bounded, say range $f \subseteq [m, M]$. Then $\underline{\int_E} f = \overline{\int_E} f$ (to be denoted by $\int_E f$) iff f is measurable.

Pf $\Leftarrow$. Let $\varepsilon > 0$. By the 2nd Principle of Littlewood (for the bounded mea. function f), $\exists$ simple φ, ψ s.t. $\varphi \leq f \leq \psi$ s.t. $|\psi - \varphi| \leq \varepsilon$ on E (why? Details pl.). Then $0 \leq \overline{\int_E} f - \underline{\int_E} f \leq \int_E \psi - \int_E \varphi = \int_E (\psi - \varphi) \leq \int_E \varepsilon = \varepsilon \cdot m(E)$.

Since ε is arbitrary & $m(E) < +\infty$ it follows that $\overline{\int_E} f = \underline{\int_E} f$.

$\Rightarrow$:

$\Rightarrow$. Take simple φ_n, ψ_n s.t. $\varphi_n \leq f \leq \psi_n$ on E s.t

$$\overline{\int_E} f + \frac{1}{2n} > \int_E \psi_n$$

$$\underline{\int_E} f - \frac{1}{2n} < \int_E \varphi_n$$

Since $\overline{\int_E} f = \underline{\int_E} f$ it follows that

$$\int_E \psi_n - \frac{1}{2n} < \overline{\int_E} f = \underline{\int_E} f < \int_E \varphi_n + \frac{1}{2n}$$

and so $0 \leq \int_E (\psi_n - \varphi_n) < \frac{1}{2n} \times 2 = \frac{1}{n}$. Noting

$\varphi_n \leq \bigvee_{n\in\mathbb{N}} \varphi_n \leq f \leq \bigwedge_{n\in\mathbb{N}} \psi_n \leq \psi_n$ on E with $\underbrace{\bigvee_{n\in\mathbb{N}} \varphi_n}_{\stackrel{\text{def}}{=} g}$, $\underbrace{\bigwedge_{n\in\mathbb{N}} \psi_n}_{\stackrel{\text{def}}{=} h}$ measurable,

it follows that $g \leq f \leq h$ a.e. on E. It remains to show that $g = h$ a.e. on E. To do this, let $\Delta \stackrel{\text{def}}{=} \{x \in E : g(x) < h(x)\}$ and $\Delta_\nu = \{x \in E : g(x) + \frac{1}{\nu} < h(x)\}$ $(\nu \in \mathbb{N})$. We have to show that $m(\Delta) = 0$ for which it suffices to show that $m(\Delta_\nu) = 0 \ \forall \nu \in \mathbb{N}$. We fix $\nu \in \mathbb{N}$. Note that

$$\Delta_\nu \subseteq \{x \in E : \varphi_n(x) + \tfrac{1}{\nu} < \psi_n(x)\} = \{x \in E : \tfrac{1}{\nu} < \psi_n(x) - \varphi_n(x)\}$$

and so, by the monotonicity and linearity,

$$\frac{1}{\nu} \cdot m(\Delta_\nu) \leq \int_{\Delta_\nu} (\psi_n - \varphi_n) \leq \int_E (\psi_n - \varphi_n) < \frac{1}{n} \quad \forall n$$

and it follows that $\frac{1}{\nu} m(\Delta_\nu) \leq 0$, i.e. $m(\Delta_\nu) = 0$, valid $\forall \nu \in \mathbb{N}$. Therefore $\Delta = \bigcup_{\nu\in\mathbb{N}} \Delta_\nu$ is of measure zero.

From now on (for this section), consider functions from the class $\overline{\mathcal{BMF}}(E)$: the class of all bounded measurable functions on E with $m(E) < +\infty$

Th2 (Linearity and Monotonicity). Let $m(E) < +\infty$

1. $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g \quad \forall \alpha, \beta \in \mathbb{R}$ & $f, g \in \mathcal{BMF}_i(E)$

2. Let A be a measurable subset of E and $f \in \mathcal{BMF}_i(E)$ (so $\int_A f$ is meaningful: directly defined for A replacing E or $\int_A f = \int_E (f \chi_A)$ — two definitions coincide).

$$\int_{A_1 \cup A_2} f = \int_{A_1} f + \int_{A_2} f \quad \forall f \in \mathcal{BMF}_i(E), \forall$$

disjoint, mea. subsets A_1, A_2 of E (so $\chi_{A_1 \cup A_2} = \chi_{A_1} + \chi_{A_2}$

3. If $f \leq g$ a.e. on E & $f, g \in \mathcal{BMF}_i(E)$ then

$\int_E f \leq \int_E g$ with equality holding true iff $f = g$ a.e on E

Pf. Assume $f \leq g$ a.e on E & $\int_E f = \int_E g$. Then $0 \leq g - f$ a.e on E and $\int_E (g - h) = 0$, and hence $\varepsilon\, m(\Delta_\varepsilon) \leq \int_{\Delta_\varepsilon} (g-h) \leq \int_E (g-h) = 0 \; \forall \varepsilon$

where $\Delta_\varepsilon := \{x \in E : \varepsilon \leq (g-h)(x)\}$. Therefore $m(\Delta_\varepsilon) = 0 \; \forall \varepsilon >$

and so

$$\Delta := \{x \in E : 0 < (g-h)(x)\} = \bigcup_{0 < \varepsilon \in \mathbb{Q}} \Delta_\varepsilon \text{ is of mea} = 0$$

Consequently $g = h$ a.e on E.

The other parts are easy exercises.

Th3. (Bounded Conv. Th). Let $f_n, f \in \mathcal{BMF}_i(E)$ with $m(E) < +\infty$ s.t. $f_n(x) \to f(x)$ ~~a.e~~ $\forall x$ in E. Suppose $\exists M \in \mathbb{R}$ s

$$|f_n(x)| \leq M \quad \forall x \in E$$

Then $\int_E f_n \to \int_E f$.

Note. "$\forall x$ in E" can be replaced by
"for a.e. x in E" throughout!

Proof. See Lecture Notes or Royden (use the 3rd Principle of Littlewood).

Generalization: $\mathcal{BMF}_0(\mathbb{R})$ (consisting of all bounded measurable functions on $\mathbb{R}$ vanishing outside some measurable set of a finite measure):
For $f \in \mathcal{BMF}_0(\mathbb{R})$ and $m(E) < +\infty$ s.t. $f = 0$ on $\mathbb{R} \setminus E$.
We define

$$\int_{\mathbb{R}} f \overset{\text{def}}{=} \int_E f$$

(the integral on E of the restriction function on f on E)

Easy to verify that this definition is well-define: if $m(A) < +\infty$ and $f = 0$ on $\mathbb{R} \setminus A$ then

$$\int_E f = \int_A f \quad \left(= \int_{E \cap A} f = \int_{E \cup A} f \right)$$

Th3' Let $f_n, f \in \mathcal{BMF}_0(\mathbb{R})$ s.t. $f_n \to f$ a.e on $\mathbb{R}$ and $|f_n| \le M$ a.e on $\mathbb{R}$ $\forall n \in \mathbb{N}$. Suppose further that $\exists E \subseteq \mathbb{R}$ with $m(E) < +\infty$ s.t

$$f_n = 0 \text{ on } \mathbb{R} \setminus E, \ \forall n \in \mathbb{N}.$$

Then $\int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f$.